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EXACT MODULI OF LAYERED PIEZOELECTRIC MEDIA

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Abstract—We derive exact results for overall moduli of multiphase piezoelectric layered media. Specific formulae are given for elastic, piezoelectric and dielectric constants together with thermal stress and pyroelectric coefficients in terms of phase moduli and volume fractions. The constituent layers and the composite aggregate are assumed to be orthorhombic of class 2 mm. The method of solution simply follows from the observation that under certain homogeneous loadings, a number of local fields are constant from layer to layer. This permits the determination of the full set of effective moduli. Proofs that guarantee that these solutions comply with the general connections of composites with cylindrical microgeometries are given. One of the possible applications can be directed to media composed of curvilinearly anisotropic layers.

1. INTRODUCTION

Piezoelectric composites have attracted wide applications in electromechanical devices and in smart material systems (Smith, 1989, 1991). The determination of the overall physical and mechanical properties of piezoelectric composites in terms of phase moduli, volume fractions and geometry is an important issue in the designing and manufacturing process. Various types of composites have been considered in the literature. For example, Chen (1994) derived simple formulae for estimates of the overall thermoelectroelastic moduli of fibrous composites (1–3 connectivity) with the self-consistent and Mori–Tanaka methods. Avellaneda and Swart (1994) calculated the effective moduli and electro-acoustic performance of 1–3 piezocomposites. Dunn and Taya (1993a, b) examined piezoelectric composites containing ellipsoidal inhomogeneities. In addition, explicit results are given for the overall moduli of multiphase platelet-reinforced composites (Chen, 1996). Layered media, the 2–2 connectivity type, are another important class of engineering materials which are extensively used in multilayer ceramic capacitors and actuators (Newham and Ruschau, 1993). The objective of this work is to derive the exact effective moduli of multiphase layered composites.

We consider a layered medium consisting of many perfectly bonded layers which are orthorhombic of class 2 mm (Nye, 1957), that is the effective law is invariant under a reflection about the x_1–x_3 and x_2–x_3 planes. We assume that the medium is constructed so that it can be regarded as macroscopically orthorhombic of the same class. No local or free edge effects are considered in the analysis. The method of solutions follows from the fact that under certain homogeneous loadings the exterior part of the stress tensor, the interior part of the strain tensor (Hill, 1972), the normal components of the electric displacement vector and the tangential components of electric field are constant throughout the medium. This constitutes spatially uniform fields from layer to layer. In particular, three of the six stress components, three of the six strain tensors and one of the three electric displacement components together with two of the three electric fields can be arbitrarily assigned. The remaining fields, only piecewise constant throughout, could be determined by rearranging the constitutive equations. The overall moduli are then obtained via the mean fields by averaging over the composite medium. Explicit solutions for the effective moduli, including nine elastic constants, five piezoelectric coefficients and three dielectric constants, together with three thermal stress tensors and one pyroelectric coefficient, are given for two different types of layered media; one consists of layers in which the material preferential axis, the x_3

axis, is placed perpendicular to the plane of the layers, the other is arranged so that it is parallel to the plane. Both results are, of course, applicable to the subclasses of orthorhombic symmetry and to pure elasticity. The solutions of the latter system are shown to exactly satisfy a number of exact connections existed for composites with cylindrical microgeometries.

A layered aggregate is known as one of the few composite geometries that are amenable to exact solutions for the effective moduli. Earlier studies have been primarily focused on the solutions of effective elastic moduli together with their thermal properties, see, for example, Postma (1955), Pagano (1974), Chou and Carleone (1974) and a recent paper by Norris (1990). The work of Grekov *et al.* (1987) seemed to be the first relevant work in examining the effective behavior of two-phase lamellar piezoelectric systems with tetragonal or higher symmetry. Related results on two-phase media were also given by Benveniste and Dvorak (1992) for transversely isotropic layers.

The plan of this work is as follows. We first review some basic equations for the considered systems. The key formulation is outlined in Section 2. Next, we present explicit results for the effective moduli of the considered layered media and examine the consistency with the exact connections. In closing we address some possible extensions of the present results.

2. PRELIMINARIES

The constitutive relation of a linearly piezoelectric medium can be written in the form (see for example, Tiersten 1969):

$$\begin{cases} \sigma_{ij} = L_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \lambda_{ij}\theta, \\ D_i = e_{ikl}\varepsilon_{kl} + \kappa_{ik}E_k - q_i\theta, \end{cases} \tag{1}$$

where σ_{ij} is the stress tensor, ε_{ij} the strain tensor, D_i the electric displacement vector, E_i the electric field and θ the uniform temperature change. L_{ijkl} are the elastic moduli measured in a constant electric field; κ_{ij} are the dielectric constants measured at constant strain; e_{ijk} are the piezoelectric constants; λ_{ij} are the linear thermal stress tensors; q_i are the pyroelectric coefficients. The material constants L, e, κ are, respectively fourth-rank, third-rank, and second-rank tensors, which satisfy the symmetry relations

$$L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij}, e_{ikl} = e_{ilk}, \kappa_{ij} = \kappa_{ji}, \tag{2}$$

so that L_{ijkl}, e_{ijk} and κ_{ij} admit, at most, twenty one, eighteen and six independent components, respectively. If u_i and ϕ are the elastic displacement vector and electric potential, the strain and electric fields are given by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), E_i = -\phi_{,i}. \tag{3}$$

It is convenient to write eqn (1) in a matrix notation. This can be achieved by using the following convention: replace the first two suffix by a single one running from 1 to 6, and the last two in the same way, according to the following correspondence

tensor notation	11	22	33	23, 32	31, 13	12, 21
matrix notation	1	2	3	4	5	6.

Accordingly, eqn (1) can be written in the form

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\varepsilon} - \mathbf{e}^{\mathrm{T}}\mathbf{E} - \boldsymbol{\lambda}\theta, \\ \mathbf{D} = \mathbf{e}\boldsymbol{\varepsilon} + \boldsymbol{\kappa}\mathbf{E} - \mathbf{q}\theta, \end{cases} \tag{4}$$

where

$$
\begin{aligned}
\sigma_m &= \sigma_{ij} \quad \text{for } m = 1,2,3,4,5,6, \quad i,j = 1,2,3,\\
\varepsilon_m &= \varepsilon_{ij} \quad \text{for } i = j, \quad m = 1,2,3,\\
\varepsilon_m &= 2\varepsilon_{ij} \quad \text{for } i \neq j, \quad m = 4,5,6,\\
L_{mn} &= L_{ijkl} \quad \text{for } i,j,k,l = 1,2,3\,; \quad m,n = 1,\ldots 6,\\
e_{in} &= e_{ikl} \quad \text{for } i,k,l = 1,2,3\,; \quad n = 1,\ldots 6,\\
\lambda_m &= \lambda_{ij} \quad \text{for } i,j = 1,2,3\,; \quad m = 1,\ldots 6.
\end{aligned} \tag{5}
$$

The moduli $\mathbf{L}$, $\mathbf{e}$, and $\boldsymbol{\kappa}$ are then represented by (6×6), (3×6), and (3×3) matrices, respectively. $\mathbf{e}^{\mathrm{T}}$ is the transpose of $\mathbf{e}$ and is a (6×3) matrix. At interfaces, the traction vector, the displacement, electric potential and the normal component of electric displacement must be continuous if perfect bonding is assumed. Alternatively, it is known (Hill, 1972) that these interface conditions are equivalent to the continuities of exterior part of stress $\boldsymbol{\sigma}_e$, interior part of strain $\boldsymbol{\varepsilon}_i$, the normal component of electric displacement $\mathbf{D}_n$ and the tangential component of electric field $\mathbf{E}_t$, in which their components are, respectively, defined as

$$
\begin{aligned}
\boldsymbol{\sigma}_e &= \mathbf{n}\otimes\boldsymbol{\sigma}\mathbf{n} + \boldsymbol{\sigma}\mathbf{n}\otimes\mathbf{n} - (\mathbf{n}\cdot\boldsymbol{\sigma}\mathbf{n})\mathbf{n}\otimes\mathbf{n},\\
\boldsymbol{\varepsilon}_i &= (\mathbf{1}-\mathbf{n}\otimes\mathbf{n})\boldsymbol{\varepsilon}(\mathbf{1}-\mathbf{n}\otimes\mathbf{n}),\\
\mathbf{D}_n &= (\mathbf{n}\otimes\mathbf{n})\mathbf{D}, \quad \mathbf{E}_t = (\mathbf{1}-\mathbf{n}\otimes\mathbf{n})\mathbf{E},
\end{aligned} \tag{6}
$$

where $\mathbf{n}$ is the unit normal of any point P on an interface surface S and $\mathbf{1}$ is the unit second-rank tensor. It is helpful to note that for a point P whose normal is along the x_3 axis, the components of $\boldsymbol{\tau}_e$ and $\boldsymbol{\tau}_i$ of a second-order tensor $\boldsymbol{\tau}$ can be expressed by

$$
\boldsymbol{\tau}_e = \begin{bmatrix} 0 & 0 & \tau_{13} \\ 0 & 0 & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}, \quad \boldsymbol{\tau}_i = \begin{bmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{12} & \tau_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}
$$

In addition, $\mathbf{v}_n$ and $\mathbf{v}_t$ are two orthogonal components of a vector $\mathbf{v}$, one perpendicular and one parallel to the surface. In view of the decomposition (6), it follows that $\boldsymbol{\tau} = \boldsymbol{\tau}_e + \boldsymbol{\tau}_i$ and $\mathbf{v} = \mathbf{v}_n + \mathbf{v}_t$.

We now partition (4) into

$$
\begin{bmatrix} \boldsymbol{\sigma}_e \\ \boldsymbol{\sigma}_i \\ \mathbf{D}_n \\ \mathbf{D}_t \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{e}_1 & \mathbf{e}_2 \\ \mathbf{L}_3 & \mathbf{L}_4 & \mathbf{e}_3 & \mathbf{e}_4 \\ \mathbf{e}_5 & \mathbf{e}_6 & \boldsymbol{\kappa}_1 & \boldsymbol{\kappa}_2 \\ \mathbf{e}_7 & \mathbf{e}_8 & \boldsymbol{\kappa}_3 & \boldsymbol{\kappa}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_e \\ \boldsymbol{\varepsilon}_i \\ \mathbf{E}_n \\ \mathbf{E}_t \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \theta, \tag{8}
$$

and rewrite (8) as

$$
\begin{aligned}
\hat{\boldsymbol{\sigma}}_e &= \hat{\mathbf{L}}_{ee}\hat{\boldsymbol{\varepsilon}}_e + \hat{\mathbf{L}}_{ei}\hat{\boldsymbol{\varepsilon}}_i - \hat{\boldsymbol{\lambda}}_e\theta,\\
\hat{\boldsymbol{\sigma}}_i &= \hat{\mathbf{L}}_{ie}\hat{\boldsymbol{\varepsilon}}_e + \hat{\mathbf{L}}_{ii}\hat{\boldsymbol{\varepsilon}}_i - \hat{\boldsymbol{\lambda}}_i\theta,
\end{aligned} \tag{9}
$$

with the definitions

$$
\hat{\boldsymbol{\sigma}}_e = \begin{Bmatrix} \boldsymbol{\sigma}_e \\ \mathbf{D}_n \end{Bmatrix}, \quad \hat{\boldsymbol{\sigma}}_i = \begin{Bmatrix} \boldsymbol{\sigma}_i \\ \mathbf{D}_t \end{Bmatrix}, \quad \hat{\boldsymbol{\varepsilon}}_e = \begin{Bmatrix} \boldsymbol{\varepsilon}_e \\ \mathbf{E}_n \end{Bmatrix}, \quad \hat{\boldsymbol{\varepsilon}}_i = \begin{Bmatrix} \boldsymbol{\varepsilon}_i \\ \mathbf{E}_t \end{Bmatrix}, \quad \hat{\boldsymbol{\lambda}}_e = \begin{Bmatrix} \boldsymbol{\lambda}_1 \\ \mathbf{q}_1 \end{Bmatrix}, \quad \hat{\boldsymbol{\lambda}}_i = \begin{Bmatrix} \boldsymbol{\lambda}_2 \\ \mathbf{q}_2 \end{Bmatrix},
$$

$$
\hat{\mathbf{L}}_{ee} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{e}_1 \\ \mathbf{e}_5 & \boldsymbol{\kappa}_1 \end{bmatrix}, \quad \hat{\mathbf{L}}_{ei} = \begin{bmatrix} \mathbf{L}_2 & \mathbf{e}_2 \\ \mathbf{e}_6 & \boldsymbol{\kappa}_2 \end{bmatrix}, \quad \hat{\mathbf{L}}_{ie} = \begin{bmatrix} \mathbf{L}_3 & \mathbf{e}_3 \\ \mathbf{e}_7 & \boldsymbol{\kappa}_3 \end{bmatrix}, \quad \hat{\mathbf{L}}_{ii} = \begin{bmatrix} \mathbf{L}_4 & \mathbf{e}_4 \\ \mathbf{e}_8 & \boldsymbol{\kappa}_4 \end{bmatrix}. \tag{10}
$$

For stable materials, $\mathbf{L}$ and $\boldsymbol{\kappa}$ are symmetric and positive definite, $\mathbf{L}_1$, $\mathbf{L}_4$, and $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_4$ are thus invertible and it follows that $\mathbf{L}_2 = \mathbf{L}_3$, $\boldsymbol{\kappa}_3 = \boldsymbol{\kappa}_4$. Naturally, the matrices $\mathbf{e}_1, \mathbf{e}_2, \ldots$ and $\mathbf{e}_8$ are not all independent. We can then rearrange (9) to get

$$\begin{bmatrix} \hat{\boldsymbol{\varepsilon}}_e \\ \hat{\boldsymbol{\sigma}}_i \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{L}}_{ee}^{-1} & -\hat{\mathbf{L}}_{ee}^{-1}\hat{\mathbf{L}}_{ei} \\ \hat{\mathbf{L}}_{ie}\hat{\mathbf{L}}_{ee}^{-1} & \hat{\mathbf{L}}_{ei} - \hat{\mathbf{L}}_{ie}\hat{\mathbf{L}}_{ee}^{-1}\hat{\mathbf{L}}_{ei} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\sigma}}_e \\ \hat{\boldsymbol{\varepsilon}}_i \end{bmatrix} - \begin{bmatrix} -\hat{\mathbf{L}}_{ee}^{-1}\boldsymbol{\lambda}_e \\ \hat{\boldsymbol{\lambda}}_i - \hat{\mathbf{L}}_{ie}\hat{\mathbf{L}}_{ee}^{-1}\hat{\boldsymbol{\lambda}}_e \end{bmatrix}\theta. \tag{11}$$

In the sequel, we shall write (11) in the form $\mathbf{s} = \Lambda\mathbf{e} + \Gamma\theta$.

3. EFFECTIVE PROPERTIES OF LAYERED MEDIA

3.1. *Layers parallel to the* x_1–x_2 *plane*

We consider a layered medium in the x_1–x_2 plane of a Cartesian coordinate consisting of several bonded layers which are constructed so that the medium can be regarded as macroscopically orthorhombic of class 2 mm. We assume that the constituent layers are also orthorhombic of the same class. The volume fraction of phase r is denoted by c_r so that $c_1 + c_2 + \cdots + c_n = 1$. The objective is to derive the exact results for the effective properties of the medium; no local or free edge effects are considered. Note that the x_3 axis is the common normal to all layers. In the first step of the formulation, each layer is subjected to a constant $\hat{\varepsilon}_i$ and $\hat{\sigma}_e$ together with a uniform temperature change θ so that ε_1, ε_2, ε_6, E_1, E_2, σ_3, σ_4, σ_5, and D_3 are uniform throughout the medium. Next, we can assemble each layer together to form an aggregate in which $\hat{\sigma}_e$ and $\hat{\varepsilon}_i$ can thus be viewed as external loads, since the interface conditions are exactly satisfied. We start from (11) to examine the effective behavior of the aggregate. By simple algebra, the volume averages of the phases are shown to be connected by

$$\bar{\hat{\boldsymbol{\varepsilon}}}_e = \frac{1}{V}\int_V \hat{\boldsymbol{\varepsilon}}_e \,\mathrm{d}v = \Sigma c_r(\bar{\hat{\boldsymbol{\varepsilon}}}_e)_r, \quad \bar{\hat{\boldsymbol{\sigma}}}_i = \frac{1}{V}\int_V \hat{\boldsymbol{\sigma}}_i \,\mathrm{d}v = \Sigma c_r(\bar{\hat{\boldsymbol{\sigma}}}_i)_r, \tag{12}$$

in which the overbar denotes the volume averaged quantities. By the definition

$$\bar{\mathbf{s}} = \Lambda\mathbf{e} + \Gamma\theta, \tag{13}$$

it follows that

$$\Lambda = \Sigma c_r \Lambda_r, \quad \Gamma = \Sigma c_r \Gamma_r. \tag{14}$$

Equation (14) is the fundamental result for the solutions of effective moduli.

We now present results for the considered system. The thermoelectroelastic moduli of an orthorhombic layer with class 2 mm are of the forms:

$$\mathbf{L} = \begin{bmatrix} L_{11} & L_{12} & L_{13} & 0 & 0 & 0 \\ L_{12} & L_{22} & L_{23} & 0 & 0 & 0 \\ L_{13} & L_{23} & L_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{66} \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{e} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\kappa} = \begin{bmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ q_3 \end{bmatrix}. \tag{15}$$

These are, in total, nine elastic constants, three dielectric permittivities and five piezoelectric constants together with three thermal stress constants and one pyroelectric coefficient. The above material constants also describe the constitutive relations for an orthorhombic medium of class mmm, in which all the piezoelectric coefficients are absent.

Now recasting (15) into the form (8), manipulating the matrices according to (11) and using the volume averaging formulae (14), after some straightforward algebra, one finds the effective constants as

$$L_{66} = \Sigma c_r L^r_{66}, \quad \frac{1}{L_{44}} = \Sigma c_r \frac{1}{L^r_{44}}, \quad \frac{1}{L_{55}} = \Sigma c_r \frac{1}{L^r_{55}}, \tag{16}$$

$$\frac{e_{24}}{L_{44}} = \Sigma c_r \frac{e^r_{24}}{L^r_{44}}, \quad \frac{e_{15}}{L_{55}} = \Sigma c_r \frac{e^r_{15}}{L^r_{55}}, \tag{17}$$

$$\kappa_{11} = \Sigma c_r \kappa^r_{11} - \frac{e^2_{15}}{L_{55}} + \Sigma c_r \frac{e^{r2}_{15}}{L^r_{55}}, \quad \kappa_{22} = \Sigma c_r \kappa^r_{22} - \frac{e^2_{24}}{L_{44}} + \Sigma c_r \frac{e^{r2}_{24}}{L^r_{44}}, \tag{18}$$

$$\frac{L_{33}}{L_{33}\kappa_{33} + e^2_{33}} = \Sigma \frac{c_r L^r_{33}}{\delta_r}, \quad \frac{e_{33}}{L_{33}\kappa_{33} + e^2_{33}} = \Sigma \frac{c_r e^r_{33}}{\delta_r}, \quad \frac{\kappa_{33}}{L_{33}\kappa_{33} + e^2_{33}} = \Sigma \frac{c_r \kappa^r_{33}}{\delta_r}, \tag{19}$$

$$L_{13} = L_{33}\Sigma c_r \frac{L^r_{13}\kappa^r_{33} + e^r_{31}e^r_{33}}{\delta_r} + e_{33}\Sigma c_r \frac{L^r_{13}e^r_{33} - e^r_{31}L^r_{33}}{\delta_r}, \tag{20}$$

$$L_{23} = L_{33}\Sigma c_r \frac{L^r_{23}\kappa^r_{33} + e^r_{32}e^r_{33}}{\delta_r} + e_{33}\Sigma c_r \frac{L^r_{23}e^r_{33} - e^r_{32}L^r_{33}}{\delta_r}, \tag{21}$$

$$e_{31} = e_{33}\Sigma c_r \frac{L^r_{13}\kappa^r_{33} + e^r_{31}e^r_{33}}{\delta_r} - \kappa_{33}\Sigma c_r \frac{L^r_{13}e^r_{33} - e^r_{31}L^r_{33}}{\delta_r}, \tag{22}$$

$$e_{32} = e_{33}\Sigma c_r \frac{L^r_{23}\kappa^r_{33} + e^r_{32}e^r_{33}}{\delta_r} - \kappa_{33}\Sigma c_r \frac{L^r_{23}e^r_{33} - e^r_{32}L^r_{33}}{\delta_r}, \tag{23}$$

$$L_{11} = \Sigma c_r L^r_{11} - \Sigma c_r (L^r_{13} - L_{13}) \frac{L^r_{13}\kappa^r_{33} + e^r_{31}e^r_{33}}{\delta_r} - \Sigma c_r (e^r_{31} - e_{31}) \frac{L^r_{13}e^r_{33} - e^r_{31}L^r_{33}}{\delta_r}, \tag{24}$$

$$L_{22} = \Sigma c_r L^r_{22} - \Sigma c_r (L^r_{23} - L_{23}) \frac{L^r_{23}\kappa^r_{33} + e^r_{32}e^r_{33}}{\delta_r} - \Sigma c_r (e^r_{32} - e_{32}) \frac{L^r_{23}e^r_{33} - e^r_{32}L^r_{33}}{\delta_3}, \tag{25}$$

$$\begin{aligned} L_{12} &= \Sigma c_r L^r_{12} - \Sigma c_r (L^r_{13} - L_{13}) \frac{L^r_{23}\kappa^r_{33} + e^r_{32}e^r_{33}}{\delta_r} - \Sigma c_r (e^r_{31} - e_{31}) \frac{L^r_{23}e^r_{33} - e^r_{32}L^r_{33}}{\delta_r} \\ &= \Sigma c_r L^r_{12} - \Sigma c_r (L^r_{23} - L_{23}) \frac{L^r_{13}\kappa^r_{33} + e^r_{31}e^r_{33}}{\delta_r} - \Sigma c_r (e^r_{32} - e_{32}) \frac{L^r_{13}e^r_{33} - e^r_{31}L^r_{33}}{\delta_r}, \end{aligned} \tag{26}$$

where $\delta_r = L^r_{33}\kappa^r_{33} + e^{r2}_{33}$.

This completes all 17 electroelastic moduli. Before we proceed to the results for thermal stress and pyroelectric coefficients, we observe that there exist a few connections between the moduli, which may offer alternative expressions for these results. First, from (19) it can be shown that

$$\left\{\Sigma\frac{c_r L^r_{33}}{\delta_r}\right\}\left\{\Sigma\frac{c_r \kappa^r_{33}}{\delta_r}\right\}+\left\{\Sigma\frac{c_r e^r_{33}}{\delta_r}\right\}^2=\frac{1}{L_{33}\kappa_{33}+e^2_{33}}. \tag{27}$$

In addition, from (20) and (22) we find the following identities

$$\Sigma c_r\frac{L^r_{13}\kappa^r_{33}+e^r_{31}e^r_{33}}{\delta_r}=\frac{L_{13}\kappa_{33}+e_{31}e_{33}}{L_{33}\kappa_{33}+e^2_{33}},\quad \Sigma c_r\frac{L^r_{13}e^r_{33}-e^r_{31}L^r_{33}}{\delta_r}=\frac{L_{13}e_{33}-e_{31}L_{33}}{L_{33}\kappa_{33}+e^2_{33}}. \tag{28}$$

Similarly, from (21) and (23) we find

$$\Sigma c_r\frac{L^r_{23}\kappa^r_{33}+e^r_{32}e^r_{33}}{\delta_r}=\frac{L_{23}\kappa_{33}+e_{32}e_{33}}{L_{33}\kappa_{33}+e^2_{33}},\quad \Sigma c_r\frac{L^r_{23}e^r_{33}-e^r_{32}L^r_{33}}{\delta_r}=\frac{L_{23}e_{33}-e_{32}L_{33}}{L_{33}\kappa_{33}+e^2_{33}}. \tag{29}$$

It is noteworthy that the effective constants L_{44}, L_{55}, L_{66} under a constant electric field take the same form as those for non-piezoelectric media. Also, it is easy to prove that, when the piezoelectric coefficients are absent, the moduli will reduce to those purely elastic solids (Postma, 1955). In addition, for the case of transversely isotropic constituents the results (16)–(26) are identical to the effective constants of platelet reinforced piezoelectric composites estimated by the self-consistent and Mori–Tanaka methods (Chen, 1996). This is an interesting and surprising outcome. This coincidence may be explained by the fact that the sharp edged effects of the platelet were neglected, and thus disc-shaped inclusion, modeled as a very thin oblate spheroid, behaves much like a thin layer.

We now turn to the effective thermal stress constants λ_1, λ_2, λ_3 and pyroelectric coefficient q_3 of the layered aggregate. Setting $\theta \neq 0$ in (14_2) yields four sets of equations expressing the effective $\boldsymbol{\lambda}$ and $\mathbf{q}$ in terms of $\mathbf{L}$, $\mathbf{e}$, $\boldsymbol{\kappa}$ and the constituent properties. Specifically, the explicit results are given by

$$\lambda_1=\Sigma c_r\lambda^r_1-\Sigma c_r(\lambda^r_3-\lambda_3)\frac{L^r_{13}\kappa^r_{33}+e^r_{31}e^r_{33}}{\delta_r}-\Sigma c_r(q^r_3-q_3)\frac{L^r_{13}e^r_{33}-e^r_{31}L^r_{33}}{\delta_r}, \tag{30}$$

$$\lambda_2=\Sigma c_r\lambda^r_2-\Sigma c_r(\lambda^r_3-\lambda_3)\frac{L^r_{23}\kappa^r_{33}+e^r_{32}e^r_{33}}{\delta_r}-\Sigma c_r(q^r_3-q_3)\frac{L^r_{23}e^r_{33}-e^r_{32}L^r_{33}}{\delta_r}, \tag{31}$$

$$\lambda_3=\Sigma c_r\frac{e^r_{33}e_{33}+\kappa^r_{33}L_{33}}{\delta_r}\lambda^r_3+\Sigma c_r\frac{e^r_{33}L_{33}-L^r_{33}e_{33}}{\delta_r}q^r_3, \tag{32}$$

$$q_3=\Sigma c_r\frac{\kappa^r_{33}e_{33}-\kappa_{33}e^r_{33}}{\delta_r}\lambda^r_3+\Sigma c_r\frac{e^r_{33}e_{33}+L^r_{33}\kappa_{33}}{\delta_r}q^r_3. \tag{33}$$

In deriving the explicit formulae, if use is made of connections (28) and (29), eqns (30) and (31) could be written in the following forms

$$\lambda_1=\Sigma c_r\lambda^r_1-\Sigma c_r\frac{\kappa^r_{33}(L^r_{31}-L_{31})+e^r_{33}(e^r_{31}-e_{31})}{\delta_r}\lambda^r_3-\Sigma c_r\frac{e^r_{33}(L^r_{31}-L_{31})-L^r_{33}(e^r_{31}-e_{31})}{\delta_r}q^r_3, \tag{34}$$

$$\lambda_2=\Sigma c_r\lambda^r_2-\Sigma c_r\frac{\kappa^r_{33}(L^r_{32}-L_{32})+e^r_{33}(e^r_{32}-e_{32})}{\delta_r}\lambda^r_3-\Sigma c_r\frac{e^r_{33}(L^r_{32}-L_{32})-L^r_{33}(e^r_{32}-e_{32})}{\delta_r}q^r_3. \tag{35}$$

Again, if constituent layers are transversely isotropic, the results (32)–(35) are exactly the same with the effective thermal stress and pyroelectric constants of platelet reinforced composites estimated by the self-consistent and Mori–Tanaka approximations (Chen, 1996). Equations (30)–(33) indicate that if the thermal stress constant λ_3^r and pyroelectric coefficient q_3^r of the phases vanish, so does the effective constants λ_3 and q_3, and in this case it turns out that $\lambda_1 = \Sigma c_r \lambda_1^r$, $\lambda_2 = \Sigma c_r \lambda_2^r$.

3.2. *Layers parallel to the* x_1–x_3 *plane*

We now consider another class of layered media in which all constituent layers are parallel to the x_1–x_3 plane and normal to the x_2 axis. The phase and overall moduli are still orthorhombic of class 2 mm as described in (15). The composite system may seem akin to that examined in Section 3.1, however the overall behavior of these two systems is not the same. The difference is that in the previous system the material preferential axis (the x_3 axis) is placed along the normal of the layers, while in the present section we consider the case in which the x_3 axis is parallel to one direction of the plane. The latter can be regarded as a special case of fibrous aggregates in which one dimension of the transverse section (the x_1 axis) is infinitely extended.

We now derive the effective properties of the system. According to the formulation given in Section 2, we may let ε_1, ε_3, ε_5, σ_2, σ_4, σ_6 and E_1, E_3, D_3 be constant throughout the medium so that the continuity conditions are fulfilled. Following the same routes outlined in Section 3.1, by some simple algebra, we obtain the exact results for the effective properties of the composite:

$$\frac{1}{L_{66}} = \Sigma c_r \frac{1}{L_{66}^r}, \quad \frac{1}{L_{22}} = \Sigma c_r \frac{1}{L_{22}^r}, \tag{36}$$

$$\begin{bmatrix} L_{55} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix} = \Sigma c_r \begin{bmatrix} L_{55} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}_r, \quad \begin{bmatrix} L_{44} & e_{24} \\ e_{24} & -\kappa_{22} \end{bmatrix}^{-1} = \Sigma c_r \begin{bmatrix} L_{44} & e_{24} \\ e_{24} & -\kappa_{22} \end{bmatrix}_r^{-1}, \tag{37}$$

$$\frac{L_{23}}{L_{22}} = \Sigma c_r \frac{L_{23}^r}{L_{22}^r}, \quad \frac{L_{12}}{L_{22}} = \Sigma c_r \frac{L_{12}^r}{L_{22}^r}, \quad \frac{e_{32}}{L_{22}} = \Sigma c_r \frac{e_{32}^r}{L_{22}^r}, \tag{38}$$

$$L_{11} = \Sigma c_r L_{11}^r + \frac{\left(\Sigma c_r \frac{L_{12}^r}{L_{22}^r}\right)^2}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{L_{12}^{r2}}{L_{22}^r}, \tag{39}$$

$$L_{33} = \Sigma c_r L_{33}^r + \frac{\left(\Sigma c_r \frac{L_{23}^r}{L_{22}^r}\right)^2}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{L_{23}^{r2}}{L_{22}^r}, \tag{40}$$

$$\kappa_{33} = \Sigma c_r \kappa_{33}^r - \frac{\left(\Sigma c_r \frac{e_{32}^r}{L_{22}^r}\right)^2}{\Sigma c_r \frac{1}{L_{22}^r}} + \Sigma c_r \frac{e_{32}^{r2}}{L_{22}^r}, \tag{41}$$

$$L_{13} = \Sigma c_r L_{13}^r + \frac{\left(\Sigma c_r \frac{L_{12}^r}{L_{22}^r}\right)\left(\Sigma c_r \frac{L_{23}^r}{L_{22}^r}\right)}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{L_{12}^r L_{23}^r}{L_{22}^r}. \tag{42}$$

$$e_{31} = \Sigma c_r e_{31}^r + \frac{\left(\Sigma c_r \frac{L_{12}^r}{L_{22}^r}\right)\left(\Sigma c_r \frac{e_{32}^r}{L_{22}^r}\right)}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{L_{12}^r e_{32}^r}{L_{22}^r}, \tag{43}$$

$$e_{33} = \Sigma c_r e_{33}^r + \frac{\left(\Sigma c_r \frac{L_{23}^r}{L_{22}^r}\right)\left(\Sigma c_r \frac{e_{32}^r}{L_{22}^r}\right)}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{L_{23}^r e_{32}^r}{L_{22}^r}. \tag{44}$$

It appears that the effective moduli (36)–(44) are quite different from those given in Section 3.1 as explained earlier. Further, it is observed that some formulae are formally similar to the exact properties of fibrous aggregates with equal phase transverse rigidities in shear (Chen, 1993, eqns 25–31). We shall investigate this further in Section 4.

We now turn to the effective thermal stress tensor and pyroelectric coefficient of the aggregate. By letting $\theta \neq 0$ in (14_2), similar to the step described before, we may express the effective $\boldsymbol{\lambda}$ and $\mathbf{q}$ in terms of $\mathbf{L}$, $\mathbf{e}$, $\boldsymbol{\kappa}$ and the constituent properties. Specifically, the nonvanishing results are derived in the forms

$$\lambda_1 = \Sigma c_r \lambda_1^r + \frac{\left(\Sigma c_r \frac{\lambda_2^r}{L_{22}^r}\right)\left(\Sigma c_r \frac{L_{12}^r}{L_{22}^r}\right)}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{L_{12}^r \lambda_2^r}{L_{22}^r}, \tag{45}$$

$$\lambda_2 = L_{22} \Sigma c_r \frac{\lambda_2^r}{L_{22}^r}, \tag{46}$$

$$\lambda_3 = \Sigma c_r \lambda_3^r + \frac{\left(\Sigma c_r \frac{\lambda_2^r}{L_{22}^r}\right)\left(\Sigma c_r \frac{L_{23}^r}{L_{22}^r}\right)}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{L_{23}^r \lambda_2^r}{L_{22}^r}, \tag{47}$$

$$q_3 = \Sigma c_r q_3^r + \frac{\left(\Sigma c_r \frac{\lambda_2^r}{L_{22}^r}\right)\left(\Sigma c_r \frac{e_{32}^r}{L_{22}^r}\right)}{\Sigma c_r \frac{1}{L_{22}^r}} - \Sigma c_r \frac{e_{32}^r \lambda_2^r}{L_{22}^r}. \tag{48}$$

All the results are derived for orthorhombic phases and can be reduced to their subclasses or to pure elasticity without difficulty. We remark that when the constituents are restricted to transversely isotropic and possess the same transverse shear moduli, then we can show that $\lambda_1 = \lambda_2$.

4. CONSISTENCY WITH EXACT CONNECTIONS OF FIBROUS AGGREGATES

As mentioned before, the system considered in Section 3.2 can be viewed as a particular case of fibrous aggregate in which the x_3 axis is the fiber direction. It is known that there exist quite a few exact connections between the effective moduli of fibrous aggregates with arbitrary transverse geometry (Chen, 1993; Benveniste, 1994a, b). It is then desirable to investigate whether our results comply with the exact connections. In the literature, most of the exact connections are established for transversely isotropic constituents about the fiber direction. For convenience, we reduce the moduli in (15) to transverse isotropy by the connections

$$L_{11} = L_{22} = k+m, \quad L_{12} = k-m, \quad L_{13} = L_{23} = l, \quad L_{33} = n, \quad L_{44} = L_{55} = p,$$
$$L_{66} = m, \quad e_{31} = e_{32}, \quad e_{15} = e_{24}, \quad \kappa_{11} = \kappa_{22}, \quad \lambda_1 = \lambda_2, \tag{49}$$

where k, l, n, m and p are Hill's (1964) elastic moduli under a constant electric field. To check the consistency between the results, we briefly recapitulate the exact relationships as follows. First, when the phases possess equal transverse shear moduli m, seven out of a total of ten overall moduli of a transversely isotropic composite can be found (Chen, 1993, eqns 25–31). The results are recorded as

$$k = \frac{\Sigma \frac{c_r k_r}{k_r+m}}{\Sigma \frac{c_r}{k_r+m}}, \quad l = \frac{\Sigma \frac{c_r l_r}{k_r+m}}{\Sigma \frac{c_r}{k_r+m}}, \quad e_{31} = \frac{\Sigma \frac{c_r e_{31}^r}{k_r+m}}{\Sigma \frac{c_r}{k_r+m}}, \tag{50}$$

$$n = \Sigma c_r n_r - \Sigma \frac{c_r l_r^2}{k_r+m} + \frac{\left[\Sigma \frac{c_r l_r}{k_r+m}\right]^2}{\Sigma \frac{c_r}{k_r+m}}, \tag{51}$$

$$e_{33} = \Sigma c_r e_{33}^r - \Sigma \frac{c_r l_r e_{31}^r}{k_r+m} + \frac{\left[\Sigma \frac{c_r e_{31}^r}{k_r+m}\right]\left[\Sigma \frac{c_r l_r}{k_r+m}\right]}{\Sigma \frac{c_r}{k_r+m}}, \tag{52}$$

$$\kappa_{33} = \Sigma c_r \kappa_{33}^r + \Sigma \frac{c_r (e_{31}^r)^2}{k_r+m} - \frac{\left[\Sigma \frac{c_r e_{31}^r}{k_r+m}\right]^2}{\Sigma \frac{c_r}{k_r+m}}, \tag{53}$$

and the overall shear modulus is certainly just m itself, the value common to all phases. Now setting $L_{66}^r = m_r = m$ in (36) and (38)–(44) and using the notations defined in (49), one immediately recognizes that the results in Section 3.2 indeed reduce to the exact moduli for a composite with equal phase shear moduli. Under the same condition, the effective thermal terms for the fibrous system were derived by Benveniste (1994a, eqn 68). For brevity, we shall not record the exact expressions here, but only remark that, by letting $m_r = m$ in (45)–(48), after some algebra, it can be shown that our results are exactly reduced to those found by Benveniste (1994a).

We now turn to the exact connections which involve the remaining three constants, namely p, e_{15} and κ_{11}. In this case, the overall symmetry of the composite may correspond at most to that of an orthorhombic crystal of class 2 mm. For convenience we can write the associated phase and overall constitutive equations in the forms:

$$\tilde{\mathbf{L}}_r = \begin{bmatrix} p & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}_r, \quad \tilde{\mathbf{L}}_x = \begin{bmatrix} L_{55} & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}, \quad \tilde{\mathbf{L}}_y = \begin{bmatrix} L_{44} & e_{24} \\ e_{24} & -\kappa_{22} \end{bmatrix}. \tag{54}$$

The following theorems are shown to be valid for multiphase piezoelectric composites with cylindrical phase boundaries (Chen, 1993, eqns 33, 47)

$$\bar{\mathbf{L}}_x(\bar{\mathbf{L}}_1, \bar{\mathbf{L}}_2, \ldots, \bar{\mathbf{L}}_n)\bar{\mathbf{L}}_y(\bar{\mathbf{L}}_1^{-1}, \bar{\mathbf{L}}_2^{-1}, \ldots, \bar{\mathbf{L}}_n^{-1}) = \mathbf{I},$$
$$\bar{\mathbf{L}}_a^{-1}\bar{\mathbf{L}}_x(\bar{\mathbf{L}}_a\bar{\mathbf{L}}_1, \bar{\mathbf{L}}_a\bar{\mathbf{L}}_2, \ldots, \bar{\mathbf{L}}_a\bar{\mathbf{L}}_n)\bar{\mathbf{L}}_b^{-1}\bar{\mathbf{L}}_y(\bar{\mathbf{L}}_b\bar{\mathbf{L}}_1^{-1}, \bar{\mathbf{L}}_b\bar{\mathbf{L}}_2^{-1}, \ldots, \bar{\mathbf{L}}_b\bar{\mathbf{L}}_n^{-1}) = \mathbf{I},$$
$$\bar{\mathbf{L}}_b^{-1}\bar{\mathbf{L}}_y(\bar{\mathbf{L}}_b\bar{\mathbf{L}}_1^{-1}, \bar{\mathbf{L}}_b\bar{\mathbf{L}}_2^{-1}, \ldots, \bar{\mathbf{L}}_b\bar{\mathbf{L}}_n^{-1})\bar{\mathbf{L}}_a^{-1}\bar{\mathbf{L}}_x(\bar{\mathbf{L}}_a\bar{\mathbf{L}}_1, \bar{\mathbf{L}}_a\bar{\mathbf{L}}_2, \ldots, \bar{\mathbf{L}}_a\bar{\mathbf{L}}_n) = \mathbf{I} \quad (55)$$

where the arguments in the brackets denote the phase properties and $\mathbf{I}$ is a unit (2×2) matrix. Now substituting (37) and (54) in (55), it is readily seen that eqns (55) are exactly satisfied. For two-phase media, one may set $\bar{\mathbf{L}}_1 = \mathbf{I}$ and $\bar{\mathbf{L}}_2 = \bar{\mathbf{L}}_a^{-1}\bar{\mathbf{L}}_b$ to obtain its corollary (Chen, 1993, eqn 48)

$$\bar{\mathbf{L}}_a^{-1}\bar{\mathbf{L}}_x(\bar{\mathbf{L}}_a, \bar{\mathbf{L}}_b)\bar{\mathbf{L}}_b^{-1}\bar{\mathbf{L}}_y(\bar{\mathbf{L}}_b, \bar{\mathbf{L}}_a) = \mathbf{I},$$
$$\bar{\mathbf{L}}_b^{-1}\bar{\mathbf{L}}_y(\bar{\mathbf{L}}_b, \bar{\mathbf{L}}_a)\bar{\mathbf{L}}_a^{-1}\bar{\mathbf{L}}_x(\bar{\mathbf{L}}_a, \bar{\mathbf{L}}_b) = \mathbf{I}. \quad (56)$$

Such connections were first found by Milgrom and Shtrikman (1989, eqn 27) and also derived by Benveniste (1994b, eqn 51) in a slightly different form via different approaches. Since (56) can be deduced from (55), naturally, the effective moduli (37) still follow the connection of (56).

Another set of exact relationship was established from the Milgrom–Shtrikman (1989) compatibility condition. The connection, given by Schulgasser (1992), Benveniste and Dvorak (1992), and Nan (1993) in the present context, takes the form

$$\begin{vmatrix} L_{55} & \kappa_{11} & e_{15} \\ L_{55}^1 & \kappa_{11}^1 & e_{15}^1 \\ L_{55}^2 & \kappa_{11}^2 & e_{15}^2 \end{vmatrix} = 0. \quad (57)$$

Again, substituting (37_1) into (57) it can be shown that (57) is satisfied.

5. NUMERICAL RESULTS

In order to illustrate the theoretical results, we perform a numerical computation for a two-phase piezoelectric layered aggregate. In practice, an interesting example of this kind is an isotropic layer combined with a poled piezoceramics (Grekov *et al.* 1987). In the numerical study, we consider a medium made of PZT-7A and Epoxy layers. The properties of the phases used for calculations are given in Dunn and Taya (1993). In the demonstration, we assume that the material preferential axis of the PZT-7A is along the perpendicular direction of the stacking plane, that is the case studied in Section 3.1. Clearly, the overall properties of the layered aggregate are transversely isotropic. We have correctly checked that the results reduce to the given properties when the volume concentration of one phase becomes zero. Typical strain coefficients defined by $d_{3i} \equiv -F_{9i}/F_{99}$ and $d_h \equiv d_{31}+d_{32}+d_{33}$ are, respectively, illustrated in Figs 1 and 2, where $\mathbf{F}$ is defined by

$$\mathbf{F} = \begin{bmatrix} \mathbf{L} & \mathbf{e}^{\mathrm{T}} \\ \mathbf{e} & -\boldsymbol{\kappa} \end{bmatrix}^{-1}. \quad (58)$$

Lastly, we remark that the results of Section 3.2 are consistent with the numerical solutions by Grekov *et al.* (1987) and Dunn and Taya (1993).

6. CONCLUDING REMARKS

We remark that the simple formulation presented above can be extended to a medium consisting of monoclinic layers. However, the resulting formulae for the effective moduli will become tedious and the algebra is much more involved. Another extension of the present results can be directed to layered aggregates in which each layer is curvilinearly

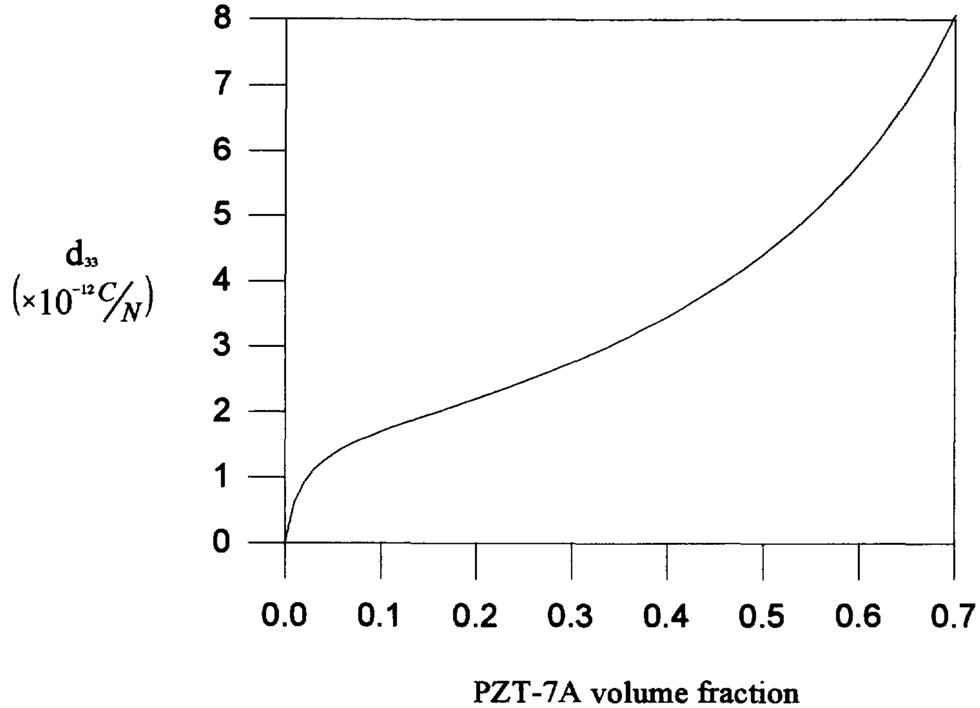

Fig. 1. Effective piezoelectric modulus d_{33} vs the volume fraction of PZT-7A.

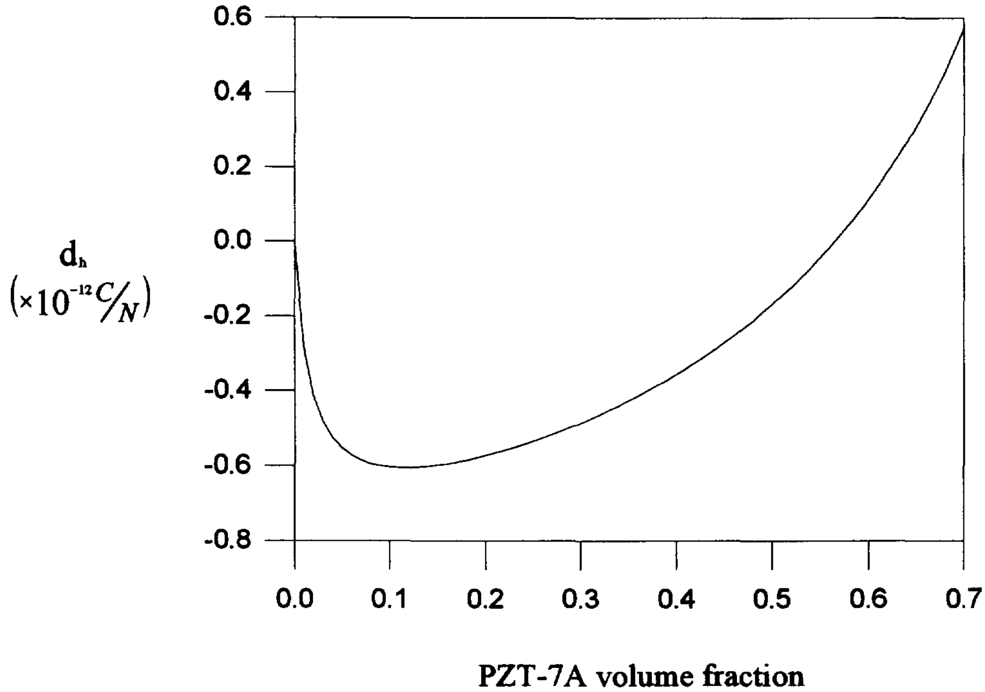

Fig. 2. Effective piezoelectric modulus d_h vs the volume fraction of PZT-7A.

anisotropic. For example, we may consider a hollow laminated cylinder constituted by many concentric layers which are cylindrically anisotropic in r, θ and z coordinates. The overall moduli of the laminated cylinder can be found in the same manner simply by letting ε_θ, ε_z, $\varepsilon_{\theta z}$, σ_r, $\sigma_{r\theta}$, σ_{rz}, E_θ, E_z and D_r be constant throughout the medium. In particular, one may directly employ the formulae given in Section 3 to obtain the results by properly locating the correspondence between x, y, z and r, θ, z. Similarly, this method is also applicable to a spherically layered medium containing spherically anisotropic (r, θ, ϕ) layers

(Christensen, 1994). In this case, the fields ε_θ, ε_ϕ, $\varepsilon_{\theta\phi}$, σ_r, $\sigma_{r\theta}$, $\sigma_{r\phi}$, E_θ, E_ϕ and D_r are assumed to be constant and the results can be directly obtained by a similar correspondence.

We finally remark that the present explicit expressions for the exact thermoelectroelastic moduli of layered piezoelectric media provide a useful tool for technological applications. In addition, they complement the existing results for piezoelectric composites and serve a check for exact relationships between the effective and phase moduli of the composite.

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